Relativistic diffusion of elementary particles with spin

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Abstract

We obtain a generalization of the relativistic diffusion of Schay and Dudley for particles with spin. The diffusion equation is a classical version of an equation for the Wigner function of an elementary particle. The elementary particle is described by a unitary irreducible representation of the Poincare group realized in the Hilbert space of wavefunctions in the momentum space. The arbitrariness of the Wigner rotation appears as a gauge freedom of the diffusion equation. The spin is described by an $SU(2)$ connection of a fiber bundle over the momentum hyperbolic space (the mass shell). Motion in an electromagnetic field, transport equations and equilibrium states are discussed.

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1. Introduction

Approximations leading to a diffusive motion seem to apply to non-relativistic as well as relativistic particles. Nevertheless, the problem of a relativistic extension of the diffusion theory encounters some difficulties. There have been various approaches to a solution of the problem (see [1–7]; for a review and further references see [8, 9]). In our earlier papers [10, 11], we followed the approach of Schay [5] and Dudley [6] who defined a relativistic analog of the Brownian motion on the phase space as the diffusion preserving the mass shell in the momentum space. We think that one should approach the relativistic dynamics starting from quantum field theory which describes multiparticle interactions in a relativistic way. In [10, 11] we have discussed the approach to equilibrium, the friction terms, the transport equations and an interaction with the electromagnetic field. In another approach, based on a relativistic Wigner function [12, 13], transport equations for relativistic particles have been derived from relativistic quantum field theory. These transport equations should have a diffusion equation as a classical non-relativistic limit.

We are interested in a classical diffusion which would approximate the relativistic motion of a quantum particle with spin. The Wigner function is the quantum analog of the phase

space distribution. The (quantum) Wigner function in general is not positively definite on the phase space although it can be positively definite on the position space or momentum space. We have shown in [11] that Schay and Dudley definition of the diffusion results from a quantum Wigner function which is positive. We have derived the relativistic diffusion of spinless particles [11] from a quantum master equation describing an evolution in the proper time (a relativistic master equation in proper time is also discussed in [14]).

In this paper we define a dissipative Lorentz invariant dynamics of elementary particles described by a positively defined probability distribution. This is an extension of the diffusion of spinless particles to particles with spin. We look for a second-order Lorentz invariant differential operator on the mass shell as a generator of the diffusion. Our starting point is an observation that the diffusion of Schay [5] and Dudley [6] as well as the diffusion of spinless massless particles [11] is generated by the operator $M_{\mu\nu}M^{\mu\nu}$ constructed from the spin-zero representation of generators $M_{\mu\nu}$ of the Lorentz group. An elementary particle is described by a unitary irreducible representation of the Poincare group. In the Hilbert space of wavefunctions defined on the momentum space, the representation is realized by an irreducible representation of the little group of Wigner rotations. There is an arbitrariness in the definition of the Wigner rotation resulting from the non-uniqueness of the boost $((m, 0, 0, 0) \rightarrow p)$. As a consequence, the generators of the Lorentz rotations depend on an arbitrary function $g(p) \in SU(2)$. The resulting diffusion equation also depends on the choice of the boost transformation. We show that the transformation of the diffusion equations can be interpreted as a gauge transformation. A gauge invariant description of the diffusion of particles with spin is possible if spin is identified with a curvature of an $SU(2)$ vector bundle over the hyperbolic space (the mass shell). The gauge invariance is used as a guiding rule for a derivation of the relativistic diffusion equation in an electromagnetic field.

In an equivalent description of spin we consider a phase space which is a product of the particle phase space with a sphere. The diffusion of a particle with spin can be described as a diffusion on the extended phase space (without any discrete variables). Then, an evolution of discrete projections σ of spin is defined by a transition function $P_\tau(x, p, \sigma; x', p', \sigma')$ which is a matrix element of the transition function on the extended phase space.

We discuss transport equations resulting from an elimination of the proper time and equilibrium distributions reached at infinite laboratory time. Applications of the spin diffusion to a description of relativistic entanglement, decoherence and magnetization are briefly discussed.

2. Wigner wavefunctions of elementary particles

Let $A \in SL(2, C)$ and Λ be a homomorphism of $SL(2, C)$ onto $SO(3, 1)$. We consider the one-particle states of a particle described by a wavefunction in the momentum space transforming under an irreducible unitary representation of the Poincare group [15–18]

$$U(A, a)\psi(p) \equiv \psi(A, a; p) = \exp(\mathrm{i}pa)D^j(A(p))\psi(\Lambda(A^{-1})p), \qquad (1)$$

where D^j is a $(2j + 1)$-dimensional irreducible unitary representation of $SU(2)$. The Wigner rotation matrix $A(p)$ in equation (1) is defined as

$$A(p) = \alpha^{-1}(p')A\alpha(p),$$

where $p' = \Lambda(A)p$ and the boost $\Lambda(\alpha(p))$ of a particle with mass m is defined by $\Lambda(\alpha(p))(mc, 0, 0, 0) = p$ (c denotes the velocity of light).

The wavefunction in the position space is defined as usual by the Fourier transform

$$\psi(x) = \int \mathrm{d}\mathbf{p} \exp(-\mathrm{i}px)\psi(p), \qquad (2)$$

where $px = p^\mu x_\mu = p_0 x_0 - \mathbf{px}$ and $p_0 = \sqrt{\mathbf{p}^2 + m^2 c^2}$ (Greek indices run from 0 to 3, Latin indices from 1 to 3 and boldface letters denote three-dimensional vectors). The Fourier transform of $U(A, a)\psi$ is a non-local function of x because $A(p)$ depends on p.

The representation (1) is unitary in the Hilbert space of functions square integrable with respect to $\mathrm{d}\mathbf{p}\, p_0^{-1}$. There are two Casimir operators characterizing irreducible representations of the Poincare group $m^2 = P^\mu P_\mu$ and the Pauli–Lubanski vector

$$w_\mu w^\mu = -m^2 j(j+1),$$

where

$$w_\mu = \tfrac{1}{2}\epsilon_{\mu\nu\sigma\rho} M^{\nu\sigma} P^\rho.$$

In the neighborhood of (1) we write

$$A = \exp\Omega,$$

where $\Omega = \mathrm{i}\omega^{\mu\nu} l_{\mu\nu}$, $\omega^{\mu\nu}$ are real parameters and $l_{\mu\nu}$ is a representation of the Lorentz algebra by 2×2 matrices. In vector notation

$$\Omega = \mathrm{i}\omega^{rs} l_{rs} + \omega^{0j} k_j = \mathrm{i}\mathbf{al} - \mathbf{vk}.$$

Here $a_j = \epsilon_{jrs}\omega_{rs}$, $l_j = \tfrac{1}{2}\epsilon_{jrs} l_{rs}$ and $v_j = -\omega_{0j}$. We define the generators $M_{\mu\nu}$ of the unitary representation of the Lorentz group by

$$U(A) = \exp(\mathrm{i}\omega^{\mu\nu} M_{\mu\nu}). \tag{3}$$

Let us note that there is some arbitrariness in the definition of the generators which is a consequence of the non-uniqueness of the boost. We can choose $\tilde{\alpha}(p) = \alpha(p)g(p)$ where $g(p)$ is an arbitrary unitary matrix. Then, the Wigner rotation is

$$\tilde{A}(p) = g^{-1}(p')A(p)g(p). \tag{4}$$

In the calculations of the generators we differentiate equation (1) over $\omega^{\mu\nu}$. Then, equation (4) leads to a gauge transformation of the generators.

First, we fix the gauge choosing (this is the conventional choice [16, 17]) α as a Hermitian matrix. In such a case, we obtain from equation (1)

$$M_{\mu\nu} = L_{\mu\nu} + \Sigma_{\mu\nu}(p), \tag{5}$$

where

$$L_{jk} = -\mathrm{i}\left(p_j \frac{\partial}{\partial p^k} - p_k \frac{\partial}{\partial p^j}\right), \tag{6}$$

$$L_{0j} = -\mathrm{i}p_0 \frac{\partial}{\partial p^j}. \tag{7}$$

$\Sigma_{\mu\nu} \in su(2)$ results from the differentiation of D in equation (1). If we write $J_l = \tfrac{1}{2}\epsilon_{lmn} M_{mn}$ and $K_l = M_{l0}$ then after differentiation of equation (1) we obtain

$$J_r = -\mathrm{i}\epsilon_{rlk} p_l \frac{\partial}{\partial p^k} + S_r, \tag{8}$$

$$K_r = \mathrm{i}p_0 \frac{\partial}{\partial p^r} + \epsilon_{rln} p_l S_n (p_0 + mc)^{-1} \equiv \mathrm{i}p_0 \frac{\partial}{\partial p^r} + \Sigma_{0r}, \tag{9}$$

where

$$[S_k, S_l] = \mathrm{i}\epsilon_{kln} S_n \tag{10}$$

is an irreducible representation of $su(2)$.

Then, the change of the boost (4) leads to a change of the generators (where $V(g)$ is the D^j representation of g appearing in equation (4)):

$$\tilde{\Sigma}_{0r} = V^{-1}\Sigma_{0r}V - V^{-1}L_{0r}V, \tag{11}$$

whereas M_{jk} does not change because for $A \in SU(2)$ the Wigner rotation $A(p)$ coincides with A.

The wavefunction (2) transforming under an irreducible representation of the Poincare group satisfies the wave equation

$$\left(c^{-2}\partial_t^2 - \nabla^2 + m^2c^2\right)\psi = 0. \tag{12}$$

We consider random superpositions of states and define the density matrix

$$\rho_{\sigma\sigma'}(p, p') = \langle\psi_\sigma(p)\overline{\psi}_{\sigma'}(p')\rangle \tag{13}$$

and

$$\rho_{\sigma\sigma'}(x, x') = \int d\mathbf{p}\, d\mathbf{p}' \exp(-ipx + ip'x')\langle\psi_\sigma(p)\overline{\psi}_{\sigma'}(p')\rangle. \tag{14}$$

Clearly, for a free motion

$$\left(c^{-2}\partial_t^2 - \nabla^2 + m^2c^2\right)\rho(x, x') = \left(c^{-2}\partial_{t'}^2 - \nabla'^2 + m^2c^2\right)\rho(x, x') = 0. \tag{15}$$

We define the Wigner function (matrix) W as the Fourier transform of ρ:

$$W_{\sigma\sigma'}(\mathbf{x}, \mathbf{p}) = \int d\mathbf{k}\, d\mathbf{k}' \int \delta\left(\mathbf{p} - \frac{1}{2}\mathbf{k} - \frac{1}{2}\mathbf{k}'\right)\exp(i(\mathbf{k} - \mathbf{k}')\mathbf{x})\rho_{\sigma\sigma'}(k, k'). \tag{16}$$

In the position space

$$W_{\sigma\sigma'}(\mathbf{x}, \mathbf{p}) = \int d\mathbf{v} \exp(-i\mathbf{p}\mathbf{v})\rho_{\sigma\sigma'}\left(\mathbf{x} + \frac{1}{2}\mathbf{v}, \mathbf{x} - \frac{1}{2}\mathbf{v}\right).$$

Note that from $\rho^+ = \rho$ it follows that

$$W^+ = W \tag{17}$$

and if $\rho \geqslant 0$ then

$$\tilde{W}(\mathbf{p}) = \int d\mathbf{x}\, W(\mathbf{x}, \mathbf{p}) = \rho(\mathbf{p}, \mathbf{p}) \geqslant 0.$$

If an observable $\Phi_{\sigma\sigma'}(\mathbf{p})$ is a function solely of $\mathbf{p}$, then the expectation value in quantum mechanics can be expressed in the form

$$\mathrm{Tr}(\rho\Phi) = \mathrm{Tr}\int d\mathbf{x}\, d\mathbf{p}\, W(\mathbf{p}, \mathbf{x})\Phi(\mathbf{p}), \tag{18}$$

where the trace on the rhs is over the spin indices. Formula (18) remains valid for observables $\Phi(\mathbf{p}, \mathbf{x})$ on the phase space. However, the Weyl ordering must be applied for a definition of a function of non-commuting operators $\mathbf{x}$ and $\mathbf{p}$.

We can also define the density matrix for a calculation of spin expectation values

$$\rho_{\sigma\sigma'} = \int d\mathbf{x}\, d\mathbf{p}\, W_{\sigma\sigma'}(\mathbf{x}, \mathbf{p}).$$

The spin matrix in relativistic quantum mechanics is often discussed in relation to the EPR experiments [19].

The normalization of the probability distribution means

$$\mathrm{Tr}\int d\mathbf{x}\, d\mathbf{p}\, W_{\sigma\sigma'}(\mathbf{x}, \mathbf{p}) = 1.$$

We treat an interaction of a particle with an environment as a motion through a random medium. We assume that the interaction preserves the Lorentz invariance. In [11] we consider the Stückelberg proper time formulation [20, 21] of quantum mechanics with a dissipation (see also [14]). In such an approach the free quantum evolution disturbed solely by a random Lorentz invariant perturbation reads

$$
\begin{aligned}
2\kappa^{-2}\big(\partial_\tau \rho + \mathrm{i}\big[\big(\partial_0^2 - \nabla^2 + m^2 c^2\big), \rho\big]\big) &= M_{\mu\nu} M^{\mu\nu} \rho + \rho M_{\mu\nu} M^{\mu\nu} - 2 M_{\mu\nu} \rho M^{\mu\nu} \\
&\quad + m^{-2} w_\mu w^\mu \rho + \rho m^{-2} w_\mu w^\mu - 2 m^{-2} w_\mu \rho w^\mu \\
&= [M_{\mu\nu}, [M^{\mu\nu}, \rho]] + m^{-2}[w_\mu, [w^\mu, \rho]].
\end{aligned}
\tag{19}
$$

Note that

$$
[M_{\mu\nu}, [M^{\mu\nu}, \rho]] = \mathbf{J}^2 \rho + \rho \mathbf{J}^2 - 2\mathbf{J}\rho\mathbf{J} - \mathbf{K}^2 \rho - \rho \mathbf{K}^2 + 2\mathbf{K}\rho\mathbf{K}.
$$

From equation (19) it follows that $\mathrm{Tr}\rho$ is preserved by the time evolution with arbitrary Hermitian operators $M_{\mu\nu}$. $\mathbf{J}$ and $\mathbf{K}$ terms enter equation (19) with different signs. The K terms are in the Lindblad form [22] whereas the J terms are not (there is a wrong sign in front of J^2). For this reason it is not clear whether the dynamics is dissipative. In section 3, it will be shown that the dynamics is well defined and dissipative if the momentum p is on the mass shell $\mathcal{H}$:

$$
p^2 = p_0^2 - p_1^2 - p_2^2 - p_3^2 = m^2 c^2.
\tag{20}
$$

In the massless case instead of $2j + 1$ states of spin j we have only two states of helicity $\lambda = j$ and $\lambda = -j$. Equations (8) and (9) should be transformed to the helicity basis before the limit $m \to 0$ [23, 24]. The formulas for generators of an irreducible representation of the Poincare group in the case $m = 0$ have been discussed in [24–26]. The different form of the generators results from different realizations of the little group $E(2)$ of massless particles. We consider the formula of Bialynicki–Birula [26] and Shirokov [25]:

$$
J_r = -\mathrm{i}\epsilon_{rlk} p_l \frac{\partial}{\partial p^k} + \frac{\lambda}{2}|\mathbf{p}|\frac{\partial}{\partial p_r} \ln(p_a p^a)
\tag{21}
$$

and

$$
K_r = \mathrm{i}|\mathbf{p}|\frac{\partial}{\partial p^r} - \lambda p_3 \epsilon_{3rb} p_b (p_a p^a)^{-1},
\tag{22}
$$

where $a = 1, 2$ and $\lambda = |\mathbf{p}|^{-1}\mathbf{p}\mathbf{J} = |\mathbf{p}|^{-1}\mathbf{p}\mathbf{S}$ is the helicity equal $\pm j$.

For later discussion we mention also the formula of Ohnuki ([18] and references quoted there):

$$
J_a = -\mathrm{i}\epsilon_{alk} p_l \frac{\partial}{\partial p^k} + \lambda p_a(|\mathbf{p}| + p_3)^{-1},
\tag{23}
$$

$$
J_3 = -\mathrm{i}\epsilon_{3lk} p_l \frac{\partial}{\partial p^k} + \lambda,
\tag{24}
$$

$$
K_a = \mathrm{i}|\mathbf{p}|\frac{\partial}{\partial p^a} + \lambda \epsilon_{3ab} p_b(|\mathbf{p}| + p_3)^{-1}
\tag{25}
$$

and

$$
K_3 = \mathrm{i}|\mathbf{p}|\frac{\partial}{\partial p^3}.
\tag{26}
$$

3. Diffusion equation

The equation for the Wigner function (16) resulting from equation (19) reads

$$\partial_\tau W = p^\mu \partial_\mu^x W + \tfrac{1}{2}\kappa^2(M_{\mu\nu}M^{\mu\nu}W + WM_{\mu\nu}M^{\mu\nu} - 2M_{\mu\nu}WM^{\mu\nu})$$
$$= p^\mu \partial_\mu^x W + \kappa^2(\mathbf{J}^2 W + W\mathbf{J}^2 - 2\mathbf{J}W\mathbf{J} - \mathbf{K}^2 W - W\mathbf{K}^2 + 2\mathbf{K}W\mathbf{K}$$
$$+ m^{-2}c^{-2}w_\mu w^\mu W + m^{-2}c^{-2}w_\mu w^\mu W - 2m^{-2}c^{-2}w_\mu W w^\mu). \tag{27}$$

The derivatives over position will have an index x, and derivatives without an index are over momenta.

The operators $\mathbf{J}$ and $\mathbf{K}$ are defined in equations (8) and (9). Writing them on the rhs in equation (27) means that the differentiation acts as usual on the function W but the multiplication by the matrix $\mathbf{S}$ is a multiplication from the right. We have for the orbital part

$$L_{\mu\nu}W(x,p) = L_{\mu\nu}(x)W + L_{\mu\nu}(p)W.$$

Unfortunately, the action of $M_{\mu\nu}$ on W is quite complicated because the spin part depends on the momenta. It is determined by a non-local integral kernel. We are unable to show whether the evolution (27) preserves the positivity of W or not. Equation (27) simplifies if we restrict ourselves to the momentum distribution

$$\tilde{W}(\mathbf{p}) = \int d\mathbf{x}\, W(\mathbf{x},\mathbf{p}) = \rho(\mathbf{p},\mathbf{p}). \tag{28}$$

In such a case, the terms $L_{\mu\nu}(x)W$ are absent because $\int d\mathbf{x}\, L_{\mu\nu}(x)W = 0$. Then, equations (27) can be rewritten as differential equations (using equations (8), (9))

$$\kappa^{-2}\partial_\tau W = \kappa^{-2}p^\mu \partial_\mu^x W + \frac{1}{2}m^2c^2\triangle_H^m W - imc(p_0+mc)^{-1}\epsilon_{rlk}p_l\frac{\partial}{\partial p^k}[S_r,W]$$
$$- \frac{1}{2}\mathbf{p}^2(p_0+mc)^{-2}[\mathbf{S},[\mathbf{S},W]] + \frac{1}{2}(p_0+mc)^{-2}[\mathbf{pS},[\mathbf{pS},W]] \equiv \mathcal{G}W, \tag{29}$$

where

$$\triangle_H^m = \partial_1^2 + \partial_2^2 + \partial_3^2 + (mc)^{-2}p_j p_k \partial^j \partial^k + 3(mc)^{-2}p_k\partial^k. \tag{30}$$

Equation (29) is equivalent to equations (19) and (27) as an equation for $\tilde{W}$ (28) (then the term $p^\mu \partial_\mu^x \tilde{W} = 0$). When $\mathbf{S} = 0$ then equation (29) coincides with the diffusion equations of Schay [5] and Dudley [6, 10, 11]. We suggest that equation (29) defines a proper formulation of the relativistic diffusion of particles with spin. Equation (29) preserves positivity (as will be shown in the subsequent sections) necessary for a probabilistic interpretation. In order to preserve the positivity we rejected in equation (29) the $L_{\mu\nu}(x)W$ terms from the dissipation part in equation (27) (such a term preserves positivity only if x is restricted to $x^2 \geqslant 0$) but we preserved the term $p^\mu \partial_\mu^x W$ as describing the spatial Hamiltonian evolution .

W is a Hermitian matrix. We could decompose it into a basis of Hermitian matrices. In the case of $j = \tfrac{1}{2}$,

$$W = W^{(0)} + W^{(k)}\sigma_k, \tag{31}$$

where σ^k are the Pauli matrices and W^μ are real functions. Equation (29) can be rewritten as a set of equations

$$\kappa^{-2}\partial_\tau W^{(0)} = \kappa^{-2}p^\mu \partial_\mu^x W^{(0)} + \tfrac{1}{2}m^2c^2\triangle_H^m W^{(0)}, \tag{32}$$

$$\kappa^{-2}\partial_\tau \mathbf{W} = \kappa^{-2}p^\mu \partial_\mu^x \mathbf{W} + \tfrac{1}{2}m^2c^2\triangle_H^m \mathbf{W} - mc(p_0+mc)^{-1}(\mathbf{p}(\partial_k W^{(k)}) - p_k\nabla W^{(k)})$$
$$- \tfrac{1}{2}\mathbf{p}^2(p_0+mc)^{-2}\mathbf{W} - \tfrac{1}{2}(p_0+mc)^{-2}(\mathbf{pW})\mathbf{p}, \tag{33}$$

where $\mathbf{W} = (W^{(1)}, W^{(2)}, W^{(3)})$ is a spin current [27] defined in general by

$$W^{(k)} = \tfrac{1}{2}\mathrm{Tr}(\sigma_k W).$$

In the massless case,

$$\rho_{\lambda\lambda'} = \langle \overline{\psi}(\lambda)\psi(\lambda')\rangle. \tag{34}$$

Here, ρ and W are 2×2 matrices. The diffusion equation reads (now $w_\mu w^\mu = 0$, we subtract the helicity in equation (27) instead of the spin; we have calculated $M_{\mu\nu}M^{\mu\nu}$ using either equations (21), (22) or (23)–(25), both give equation (35))

$$\kappa^{-2}\big(\partial_\tau W - p^\mu \partial_\mu^x W\big) = \tfrac{1}{2}\triangle_H W \tag{35}$$

with

$$\triangle_H = p_j p_k \partial^j \partial^k + 3p^k \partial_k, \tag{36}$$

where $k = 1, 2, 3$ and $\partial^j = \frac{\partial}{\partial p_j}$. We note that the generator (36) [11] is the limit $m \to 0$ of $m^2 c^2 \triangle_H$ of [10]. The diffusion equation (35) is the limit $m \to 0$ of the diffusion equation (29) if we set $\mathbf{S} \to |\mathbf{p}|^{-1}\mathbf{p}\mathbf{S}$. Equation (35) shows that the dissipation does not depend on the polarization. Its inclusion in equations (34) and (35) is superfluous; the Lorentz invariant dissipation does not depend on the polarization in the limit $m \to 0$.

4. Gauge invariance

Let us compare the operator on the rhs of equation (29) with the Laplace–Beltrami operator on an $SU(2)$ vector bundle with a connection $\mathbf{A}$ over the hyperbolic manifold (20):

$$\triangle_A = g^{-\frac{1}{2}}(\partial_j - \mathrm{i}A_j)g^{\frac{1}{2}}g^{jl}(\partial_l - \mathrm{i}A_l), \tag{37}$$

where [10]

$$g^{jl} = \delta^{jl} + m^{-2}c^{-2}p^j p^l, \tag{38}$$

$$g_{jl} = \delta^{jl} - p_0^{-2}p^j p^l \tag{39}$$

and $g = m^2 p_0^{-2}$.

We write the diffusion equation as

$$2m^{-2}\kappa^{-2}c^{-2}\big(\partial_\tau W - p^\mu \partial_\mu^x W\big) = \triangle_A W \tag{40}$$

with

$$\mathbf{A} = \frac{1}{mc(mc + p_0)}\mathbf{p} \times \mathbf{S}^{ad} \tag{41}$$

and

$$A_k g^{kl} A_l = \mathbf{p}^2 m^{-2}c^{-2}(p_0 + mc)^{-2}(\mathbf{S}^{ad}\mathbf{S}^{ad} - \mathbf{p}^{-2}\mathbf{p}\mathbf{S}^{ad}\mathbf{p}\mathbf{S}^{ad}), \tag{42}$$

where

$$\mathbf{S}^{ad}\Phi = [\mathbf{S}, \Phi].$$

A similar connection in the momentum space comes from Foldy–Wouthuysen transformation in [28] where it is applied to a description of the spin Hall effect.

The diffusion equation (33) for the spin current $\mathbf{W}$ takes the gauge invariant form (40):

$$2m^{-2}\kappa^{-2}c^{-2}\big(\partial_\tau \mathbf{W} - p^\mu \partial_\mu^x \mathbf{W}\big) = \triangle_A \mathbf{W}. \tag{43}$$

Here, the covariant derivative is $\partial_j + A_j$ with an $SO(3)$ (antisymmetric) connection:

$$A_j^{ab} = \frac{1}{mc(mc + p_0)}(p_b\delta_{ja} - p_a\delta_{jb}), \tag{44}$$

where $(A_j W)^a = A_j^{ab} W^{(b)}$ is a multiplication of the vector **W** by a matrix A. Under the gauge transformation,

$$\tilde{A}_j = O^{-1}A_j O + O^{-1}\partial_j O$$

and $\mathbf{W} \to O^{-1}\mathbf{W}$ so that equation (43) stays invariant.

The diffusion equation (29) defined by $M_{\mu\nu}M^{\mu\nu}$ depends on the choice of the generators in the Wigner representation (arbitrariness of the Wigner rotation $A(p)$ resulting from the non-uniqueness of the boost). The arbitrariness of the generators is expressed by the gauge transformation of the connection **A**:

$$\tilde{A}_j = V^{-1}A_j V + \mathrm{i}V^{-1}\partial_j V. \tag{45}$$

The gauge (41) is distinguished by the transversality condition

$$g^{jk}\partial_j A_k = 0.$$

Note that the gauge transformation (45) is different from the one of equation (11) by a factor of p_0. We would get the gauge transformation (11) if we assumed the diffusion generator in the form $M_{0j}M_{0j}$. The diffusion would have the gauge invariance (11) but no relativistic invariance. The subtraction of $M_{jk}M_{jk}$ changes the gauge transformation. The change comes from the replacement of the $p_0^2\partial_j\partial_j$ term by $m^2c^2\partial_j\partial_j$ resulting from the subtraction of $M_{jk}M_{jk}$ from $M_{0j}M_{0j}$. Clearly, this is also necessary for the Lorentz invariance.

If in equation (40) we perform the transformation (45) of the connection then after the transformation the solution of equation (40) is rotated into $V^{-1}W_\tau V$. A calculation of expectation values depends on this gauge rotation. Only gauge-independent expectation values have a physical meaning. The expectation value is expressed by the trace (18). Hence, as local observables Φ, we should consider either a multiple of an identity matrix or a local function of

$$R_{jk} = \partial_j A_k - \partial_k A_j + \mathrm{i}[A_j, A_k] \tag{46}$$

and its covariant derivatives. As non-local observables we could admit polynomials of the Wilson loop variables:

$$\Phi_C = \prod_C \exp(\mathrm{i}A_j \mathrm{d}p_j), \tag{47}$$

where the rhs denotes a product integral around a closed curve C.

5. Feynman–Kac–Ito formula

We derive a probabilistic solution of equation (29). Let us define the path-ordered phase factor (a matrix) as a solution of the equation

$$\mathrm{d}T_\tau^{ad} = \mathrm{i}A_l \circ \mathrm{d}p_\tau^l T_\tau^{ad}, \tag{48}$$

where $\mathbf{p}_\tau$ is the stochastic process on the mass shell (20) discussed in [10] (the circle denotes the Stratonovitch integral [29]). We could describe the stochastic process as the diffusion process solving the diffusion equation (32):

$$W_\tau^{(0)}(x, \mathbf{p}) = E[W^{(0)}(x_\tau, \mathbf{p}_\tau)],$$

where the expectation value $E[\cdots]$ is over the diffusion process $(x_\tau, \mathbf{p}_\tau)$ as has been discussed in detail in [10]. The solution T_τ of equation (48) is a product integral (a unitary matrix) [30]:

$$T_\tau^{ad} = \prod_{s=0}^{s=\tau} \exp\big(\mathrm{i} A_j^{ad} \circ \mathrm{d} p_s^j\big). \tag{49}$$

Then, the solution of the matrix equation (29) reads

$$W_\tau(x, \mathbf{p}) = E\big[T_\tau^{ad} W(x_\tau, \mathbf{p}(\tau))\big] = E\big[T_\tau W(x_\tau, \mathbf{p}_\tau) T_\tau^{-1}\big], \tag{50}$$

where in the last formula the matrices are in the irreducible representation (10) of $su(2)$, i.e.

$$\mathbf{A} = \frac{1}{mc(mc + p_0)} \mathbf{p} \times \mathbf{S}.$$

The vector equation (43) has the solution

$$\mathbf{W}_\tau(x, \mathbf{p}) = E[T_\tau \mathbf{W}(x_\tau, \mathbf{p}_\tau)], \tag{51}$$

where $T_\tau \in O(3)$ is the product integral with the connection (44). Equations (50) and (51) give the solutions of the diffusion equations (32) and (33).

As a consequence we obtain the diamagnetic inequality

$$|W_\tau(x, \mathbf{p})| \leqslant E[|W(x_\tau, \mathbf{p}(\tau))|],$$

i.e. the probability distribution of particles with a spin is bounded by the probability distribution of spinless particles discussed in [10]. From equation (50) it follows that if the initial value W is a Hermitian positive definite matrix then W_τ is also a Hermitian positive definite matrix, i.e.

$$\int \mathrm{d}\mathbf{p} \overline{f}_\sigma(\mathbf{p}) W_{\sigma\sigma'} f_{\sigma'}(\mathbf{p}) \geqslant 0.$$

Hence, W can be given a probabilistic interpretation which is preserved in time.

As discussed in section 4 the expectation values of $\mathbf{S}$ are not gauge invariant. For this reason $\mathbf{S}$ in not an appropriate variable for a spin. We should use

$$\hat{S}_j = 2m^2 c^2 \epsilon_{jkl} R_{kl}. \tag{52}$$

Then, the expectation value of $\hat{\mathbf{S}}$ is

$$\langle \hat{S}_j \rangle_\tau = 2m^2 c^2 \epsilon_{jkl} \mathrm{Tr} \int \mathrm{d}\mathbf{p} E\big[T_\tau^{-1} R_{kl}(\mathbf{p}) T_\tau W(\mathbf{p}_\tau(\mathbf{p}))\big].$$

It follows that the expectation value is gauge invariant. It can also be seen from equation (50) that (in the absence of an electromagnetic field) the spin evolution is just a spin rotation determined by the connection $\mathbf{A}$ (41). By differentiation of T_τ in $\langle \hat{S}_j \rangle_\tau$ using (48) and the Ito formula [29], we obtain (see also equation (29)) the double commutator

$$\partial_\tau \langle \hat{S}_j \rangle_\tau \simeq [\mathbf{S}, [\mathbf{S}, \langle \hat{S}_j \rangle_\tau]],$$

where $\mathbf{S}$ is the matrix representation of $su(2)$. Such a dissipative spin dynamics is discussed in [31].

6. Interaction with an electromagnetic field

Without external fields the spin is completely described by the Pauli–Lubanski vector (in the particle's rest frame $w = (0, m\mathbf{s})$). It does not change in time. In an electromagnetic field the amplitudes do not satisfy the Klein–Gordon equation (12). The standard way to include the interaction with an external electromagnetic field is to define bispinor amplitudes which satisfy the Dirac equation (the spacetime derivatives replaced by covariant derivatives) . The bispinor

amplitudes describe both a particle and the antiparticle. In an external field the division of Dirac wavefunctions into positive and negative frequencies (electrons and positrons) is field dependent and non-covariant. We need to eliminate the negative frequencies. We do it in an expansion in c^{-1}. Then, the gauge invariance in momentum space will be our guiding principle for a derivation of the final relativistic diffusion equation. We express the lower bispinor by the upper one. Then, we expand the field-dependent parts of the Dirac equation in powers of c^{-1}. In the lowest non-trivial order, the square of the Dirac operator for a particle in an electromagnetic vector potential **a** (determining the magnetic field $\mathbf{B} = \nabla \times \mathbf{a}$) and electric field **E** can be expressed in the form [32, 33]

$$D^2 = \partial_0^2 - \nabla_k \nabla_k + \frac{e}{mc} \mathbf{SB}, \tag{53}$$

where

$$\nabla_k = \partial_k - \mathrm{i} a_k + \frac{\mathrm{i}}{4mc^2} \epsilon_{kjl} E_j S_l.$$

As discussed in [32, 33] we obtain an extended $U(1) \times SU(2)$ gauge invariance in the position space as a result of the coupling to the spin degrees of freedom. The Pauli equation is expressed as the Klein–Gordon equation in a non-Abelian gauge field with the symmetry group $U(1) \times SU(2)$.

In the proper time approach to the density matrix evolution we insert the Pauli approximation (53) of the square D^2 of the Dirac operator in the commutator with the density matrix in equation (19). Then, the Wigner function evolution is determined by the motion of position and momenta determined by the equations

$$\frac{\mathrm{d}x^\mu}{\mathrm{d}\tau} = \frac{p^\mu}{m}, \qquad \frac{\mathrm{d}p_j}{\mathrm{d}\tau} = \frac{e}{mc} F_{j\mu} p^\mu. \tag{54}$$

The equation for the spin comes from the commutator $[D^2, \rho]$ in the master equation (19). In the classical non-relativistic limit, this equation reads (see the discussion in [34–37] and a generalization to non-Abelian gauge fields in [38–40])

$$\frac{\mathrm{d}S_j}{\mathrm{d}\tau} = \frac{e}{mc} \epsilon_{jkl} \left(B_k - \frac{1}{2mc} \epsilon_{krn} p_r E_n \right) S_l. \tag{55}$$

In the approximation (53)–(55) to equation (19), equation (29) for the evolution of the Wigner function in an electromagnetic fields takes the form

$$\kappa^{-2} \partial_\tau W = \kappa^{-2} p^\mu \partial_\mu^x W + \frac{1}{2} m^2 c^2 \triangle_H^m W - \mathrm{i}mc(p_0 + mc)^{-1}(\mathbf{p} \times \nabla)[\mathbf{S}, W]$$

$$- \frac{1}{2} \mathbf{p}^2 (p_0 + mc)^{-2}[\mathbf{S}, [\mathbf{S}, W]] + \frac{1}{2} (p_0 + mc)^{-2}[\mathbf{pS}, [\mathbf{pS}, W]]$$

$$+ \kappa^{-2} \frac{\mathrm{i}e}{mc} \left(\mathbf{B} - \frac{1}{2mc} \mathbf{p} \times \mathbf{E} \right)[\mathbf{S}, W] + \kappa^{-2} \frac{e}{m} \left(\mathbf{E}\nabla + \frac{1}{mc} \mathbf{B}(\mathbf{p} \times \nabla) \right) W. \tag{56}$$

We need to add terms of higher orders in c^{-1} in order to obtain an equation which will be gauge invariant in the momentum space and have the correct Thomas precession [34, 36, 37] of the spin (it is surprising that the $U(1) \times SU(2)$ gauge symmetry in the position space contributes to the gauge invariance in the momentum space, see also [28]).

We suggest that the gauge invariant version of equation (56) with the correct classical Thomas precession is

$$\partial_\tau W - p^\mu \partial_\mu^x W = \frac{\kappa^2}{2} m^2 c^2 \triangle_A W + \frac{\mathrm{i}e}{2m^3 c^3} \left(\mathbf{B} - \frac{p_0}{mc(mc + p_0)} \mathbf{p} \times \mathbf{E} \right)[\hat{\mathbf{S}}, W]$$

$$+ \frac{e}{mc} F_{j\mu} p^\mu (\partial_j - \mathrm{i}A_j) W. \tag{57}$$

In equation (57) the gauge covariant spin $\hat{\mathbf{S}}$ (52) has been introduced. In the lowest orders in c^{-1} only its non-relativistic version $\mathbf{S}$ (as in equation (56)) appears. The term $p_0(p_0 + mc)^{-1}$ comes from the relativistic theory of spin precession [34–37] (replacing $\frac{1}{2}$ of equation (56)).

7. Spin diffusion

In this section we show that equations (29)–(33) can be equivalently treated as equations for a function $W(x, p, \mathbf{n})$ defined on an extended phase space $R^4 \times \mathcal{H} \times S^2$ ($\mathbf{n} \in S^2$). Let $g \in SU(2)$ be represented in the form

$$g(\phi, \theta, \alpha) = g_3(\phi) g_2(\theta) g_3(\alpha), \tag{58}$$

where g_3 corresponds (under the $SU(2) \to O(3)$ homomorphism) to the rotation with respect to the third axis and g_2 corresponds to the rotation around the second axis. We can represent a point $\mathbf{n} \in S^2$ as $\mathbf{n}(\phi, \theta) = O(g(\phi, \theta, \alpha))\mathbf{n_0}$ where $\mathbf{n}_0$ is parallel to the third axis and $O \in SO(3)$. Define

$$Y_{j\sigma}(\theta, \phi) = D^j_{\sigma 0}(g(\phi, \theta, \alpha)). \tag{59}$$

For natural j these are the standard spherical functions. Let W be the Wigner $(2j+1) \times (2j+1)$ matrix (16). We define a real function on the extended phase space:

$$W(x, \mathbf{p}, \mathbf{n}) = \sum_{\sigma\sigma'} \overline{Y}_{j\sigma}(\phi, \theta) W_{\sigma\sigma'}(x, \mathbf{p}) Y_{j\sigma'}(\phi, \theta). \tag{60}$$

In the simplest case $j = \frac{1}{2}$ it can be checked by elementary calculations that if W is of the form (31) then the transformation (60) gives

$$W = W^{(0)} + n_k W^{(k)}, \tag{61}$$

where $n_3 = \cos\theta, n_1 = \sin\theta\cos\phi, n_2 = \sin\theta\sin\phi$. Hence, the expansion (61) in n_k is equivalent to the expansion (31) in the basis of Pauli matrices σ_k. In the representation (60) the spin operator is represented as a differential operator

$$S_k = -\mathrm{i}\epsilon_{klr} n^l \frac{\partial}{\partial n^r}, \tag{62}$$

where $\mathbf{n}^2 = 1$. The operators $\mathbf{S}$ of equation (62) are the generators of rotations on the unit sphere. They could be represented in the spherical coordinates (θ, ϕ) in the form well known from the theory of angular momentum in quantum mechanics.

The gauge transformation (11) is just a rotation of $\mathbf{n}$, i.e. with another realization of the Wigner rotation, when the solution of the matrix diffusion equation (29) transforms as

$$W' = V(g)^{-1} W V(g), \tag{63}$$

we have

$$W'(\mathbf{p}, \mathbf{n}) = W(\mathbf{p}, O(g)\mathbf{n}), \tag{64}$$

where $O(g)$ is the rotation corresponding to the element $g \in SU(2)$ (it may depend on $\mathbf{p}$).

We can treat equation (29) as a diffusion equation on the extended phase space. The spin evolution (55) can equivalently be described as an evolution of a point on the unit sphere:

$$\frac{\mathrm{d}\mathbf{n}}{\mathrm{d}\tau} = \frac{e}{mc} \left(\mathbf{B} \times \mathbf{n} - \frac{1}{2mc}(\mathbf{p} \times \mathbf{E}) \times \mathbf{n} \right). \tag{65}$$

It leads to the drift

$$Y^S W = \frac{e}{mc} \left(\mathbf{B} - \frac{1}{2mc} \mathbf{p} \times \mathbf{E} \right) (\mathbf{n} \times \nabla_{\mathbf{n}}) W$$

in the diffusion equation. The spin diffusion part

$$\triangle_S = \mathbf{p}^2 m^{-2} c^{-2} (p_0 + mc)^{-2} (\mathbf{S}^{ad} \mathbf{S}^{ad} - \mathbf{p}^{-2} \mathbf{p} \mathbf{S}^{ad} \mathbf{p} \mathbf{S}^{ad}) \tag{66}$$

is expressed as a differential operator

$$\triangle_S = \mathbf{p}^2 m^{-2} c^{-2} (p_0 + mc)^{-2} ((\mathbf{n} \times \nabla_{\mathbf{n}})^2 - \mathbf{p}^{-2} (\mathbf{p}(\mathbf{n} \times \nabla_{\mathbf{n}}))^2). \tag{67}$$

The term

$$- \mathrm{i} mc (p_0 + mc)^{-1} \epsilon_{rlk} p_l \frac{\partial}{\partial p^k} [S_r, W] = mc (p_0 + mc)^{-1} (\mathbf{n} \times \nabla_{\mathbf{n}}) (\mathbf{p} \times \nabla) W \tag{68}$$

describes the spin–orbit interaction. Equations (29) and (66)–(68) altogether define a diffusion on the product of the particle phase space with a unit sphere.

The complete diffusion equation of the particle with spin in magnetic and electric fields equivalent to equation (56) (with a non-relativistic spin precession) reads

$$\kappa^{-2} \partial_\tau W = \kappa^{-2} p^\mu \partial_\mu^x W + \frac{1}{2} m^2 c^2 \triangle_H^m W + \kappa^{-2} \frac{e}{mc} \left(\mathbf{B} - \frac{1}{2mc} \mathbf{p} \times \mathbf{E} \right) (\mathbf{n} \times \nabla_{\mathbf{n}}) W$$

$$+ \kappa^{-2} \frac{e}{mc} (\mathbf{p} \times \mathbf{B} + mc\mathbf{E}) \nabla W + mc(p_0 + mc)^{-1} (\mathbf{n} \times \nabla_{\mathbf{n}}) (\mathbf{p} \times \nabla) W$$

$$+ \frac{\mathbf{p}^2}{2m^2 c^2 (p_0 + mc)^2} ((\mathbf{n} \times \nabla_{\mathbf{n}})^2 - \mathbf{p}^{-2} (\mathbf{p}(\mathbf{n} \times \nabla_{\mathbf{n}}))^2) W. \tag{69}$$

In section 9 we still add a friction drift term to equation (69) which drives the diffusion to an equilibrium. A solution $W_\tau(x, \mathbf{p}, \mathbf{n})$ of the diffusion equation (69) starting from a non-negative initial condition W remains non-negative. We can relate the function W to the matrix $W_{\sigma\sigma'}$ (defined as a Fourier transform of the density matrix (16)) using expansions (31) and (60). On the extended phase space, the expectation values are defined as in classical mechanics:

$$\langle \Phi(\mathbf{p}, \mathbf{n}) \rangle_\tau = \int \mathrm{d}\mathbf{p} \, \mathrm{d}\mathbf{n} W_\tau(\mathbf{p}, \mathbf{n}) \Phi(\mathbf{p}, \mathbf{n}), \tag{70}$$

where $\mathrm{d}\mathbf{n}$ is the invariant measure on the sphere. The gauge invariance means the rotation invariance of the integral (70), i.e. it should not depend on $O(\mathbf{p})$ if we make a replacement $\mathbf{n} \to O(\mathbf{p})\mathbf{n}$.

Equation (69) is non-trivial even in the non-relativistic limit:

$$\kappa^{-2} \partial_t W = \kappa^{-2} \mathbf{p} \nabla_{\mathbf{x}} W + \frac{1}{2} m^2 c^2 \triangle W + \frac{1}{8} \mathbf{p}^2 m^{-4} c^{-4} ((\mathbf{n} \times \nabla_{\mathbf{n}})^2 - \mathbf{p}^{-2} (\mathbf{p}(\mathbf{n} \times \nabla_{\mathbf{n}}))^2) W$$

$$+ \frac{1}{2} (\mathbf{n} \times \nabla_{\mathbf{n}}) (\mathbf{p} \times \nabla)) W + \frac{e}{mc\kappa^2} \mathbf{B} (\mathbf{n} \times \nabla_{\mathbf{n}}) W$$

$$- \frac{e}{2m^2 c^2 \kappa^2} (\mathbf{p} \times \mathbf{E}) (\mathbf{n} \times \nabla_{\mathbf{n}}) W + \frac{e}{mc\kappa^2} \mathbf{p} \times \mathbf{B} \nabla W + \frac{e}{m\kappa^2} \mathbf{E} \nabla W, \tag{71}$$

where $\triangle = \partial_1^2 + \partial_2^2 + \partial_3^2$.

8. Evolution of observables

The mean values in quantum mechanics are defined in equation (18) with W as the Wigner function. We extend this definition to the diffusion

$$\langle \Phi \rangle_\tau = \mathrm{Tr} \int \mathrm{d}x \, \mathrm{d}\mathbf{p} W_\tau(\mathbf{p}, x) \Phi(\mathbf{p}, x) \equiv \mathrm{Tr} \int \mathrm{d}x \, \mathrm{d}\mathbf{p} W(\mathbf{p}, x) \Phi_\tau(\mathbf{p}, x). \tag{72}$$

If the observable Φ does not depend on x then the formula (72) coincides with the one of quantum mechanics defining the expectation values by the trace over the density matrix (the lhs

of equation (18)). Equation (72) defines expectation values in the theory of diffusion processes. Φ_τ on the rhs of equation (72) is the definition of the time evolution of Φ (the adjoint of the operator $W \to W_\tau$). We suggest that quantum expectation values are approximated by the expectation values (72) taken over the relativistic diffusion process.

As discussed in section 2, the form of the diffusion equation depends on the choice of the Wigner rotation. The expectation values (72) will not depend on the choice of the Wigner rotation if the observable Φ is covariant under the gauge transformation (45). This means that Φ should either be a scalar or transform as $V^{-1}\Phi V$ under the gauge transformation. In the latter case Φ can be a function of the curvature R (and its covariant derivatives) (46) or the loop Φ_C (47). Hence, we must work with variables $(x, \mathbf{p}, \hat{\mathbf{S}})$ (52) instead of $(x, \mathbf{p}, \mathbf{S})$. The difference appears only in the relativistic domain because if $|\mathbf{p}| << mc$ then

$$R_{jk} \simeq (2m^2c^2)^{-1}\epsilon_{jkl}S_l = (2m^2c^2)^{-1}\Sigma_{jk}.$$

If we choose the boost (4) with the rotation g then the solution of the modified diffusion equation will be $V(g)^{-1}W_\tau V(g)$ (where W_τ is the solution of the diffusion equation (29) with the Hermitian boost). As an example, according to equations (45) and (46), the gauge transformed observable $\Phi(\tilde{\hat{\mathbf{S}}})$ being a local function of $\hat{\mathbf{S}}$ is transformed into $V^{-1}\Phi(\hat{\mathbf{S}})V$. Hence, the trace in equation (72) does not depend on the choice of the Wigner rotation.

From equation (50) it follows that the diffusion with the spin is just a unitary rotation of the spin followed by the spinless evolution on the phase space, i.e.

$$\langle \Phi_\tau \rangle = \int d\mathbf{p}\, dx\, E[T_\tau^{ad} W(\mathbf{p}_\tau, x_\tau)\Phi(\mathbf{p}, x)]$$
$$= \int dx\, d\mathbf{p}\, E[T_\tau^{-1}\Phi(\mathbf{p}, x)T_\tau W(\mathbf{p}_\tau, x_\tau)], \tag{73}$$

where $(\mathbf{p}_\tau, x_\tau)$ is the solution of the same diffusion equations as discussed earlier [10] in the model without the spin.

The evolution Φ_τ of Φ in equation (72) is determined by the adjoint operator $\mathcal{G}^*$:

$$\kappa^{-2}\partial_\tau\Phi \equiv \mathcal{G}^*\Phi = -p^\mu\partial_\mu^x\Phi + \frac{1}{2}m^2\triangle_H^{m*}\Phi + im\frac{\partial}{\partial p^k}(p_0 + mc)^{-1}\epsilon_{rlk}p_l[S_r, \Phi]$$
$$- \frac{1}{2}\mathbf{p}^2(p_0 + mc)^{-2}[\mathbf{S}, [\mathbf{S}, \Phi]] + \frac{1}{2}(p_0 + mc)^{-2}[\mathbf{p}\mathbf{S}, [\mathbf{p}\mathbf{S}, \Phi]], \tag{74}$$

where

$$\triangle_H^{m*} = \partial_1^2 + \partial_2^2 + \partial_3^2 + (mc)^{-2}\partial^j\partial^k p_j p_k - 3(mc)^{-2}\partial_k p^k. \tag{75}$$

We have two candidates for time in the relativistic diffusion: τ (interpreted as the proper time) and x_0. In the non-relativistic limit, when $p_0 \simeq mc$, x_0 and τ enter the solutions of diffusion equations in an additive way. Hence, x_0 is just a shift of τ. In the relativistic case in order to obtain a solution of the diffusion equation as a function of the laboratory time x_0 we let $\tau \to \infty$ (see the discussion in [10, 11]). In the limit, we obtain solutions of equation (74) which are independent of τ. The solution of equation (29) (or more general equation (57), expressed as $\partial_\tau W = \mathcal{G}W$) does not depend on τ if

$$\mathcal{G}W = 0. \tag{76}$$

Φ does not depend on τ if

$$\mathcal{G}^*\Phi = 0. \tag{77}$$

Equation (76) is a transport equation in the laboratory time x_0 which is well-defined for $x_0 \leqslant 0$ (because it is of the form $\partial_0 W = -\triangle W$ where $-\triangle$ is a positively definite operator).

Equation (77) is a well-defined transport equation for $x_0 \geqslant 0$ (because it is of the form $\partial_0 \phi = \triangle \phi$). If either of equations (76) and (77) is satisfied, then the expectation value (72) does not depend on τ. We treat equation (77) as the basic formula determining the evolution of observables measured in experiments. Solutions of equations (77) and (57) can be related by a random change of time which is a generalization of the transformation between the evolution in proper time and laboratory time x_0 well known from the relativistic classical mechanics. In [11] we have shown that from solutions of equation (57) (without spin) by a random change of time [29] we obtain solutions of the transport equation (77). In [41] the problem is discussed in a mathematically equivalent form but in the inverse direction: from the solutions of the transport equation (77) by a random change of time the authors derive the solution of the evolution equation in the proper time.

9. An approach to the equilibrium

We are interested whether the solutions of equation (57), (74), (76) or (77) tend to a limit (the equilibrium) at $\tau \to \infty$ and $x_0 \to \pm\infty$. In order to achieve the equilibrium additional drift terms (a friction) must be added to the diffusion equation. We add a drift

$$Y = R_j \frac{\partial}{\partial p_j} \tag{78}$$

to the diffusion (29). Then

$$\mathcal{G}^Y = \mathcal{G} + Y = \tfrac{1}{2}\kappa^2 m^2 c^2 \triangle_A + Y. \tag{79}$$

Now, the diffusion equation reads

$$\partial_\tau W_\tau = \mathcal{G}^Y W_\tau. \tag{80}$$

The equation for Φ is

$$\partial_\tau \Phi_\tau = \mathcal{G}^{Y*}\Phi_\tau. \tag{81}$$

The transport equation (77) reads

$$\kappa^{-2} p_0 \partial_0 \Phi = \kappa^{-2} \mathbf{p} \nabla_\mathbf{x} \Phi + \frac{1}{2} m^2 c^2 \triangle_H^{m*} \Phi - \mathrm{i} m c \frac{\partial}{\partial p^k}(p_0 + mc)^{-1} \epsilon_{rlk} p_l [S_r, \Phi]$$

$$- \frac{1}{2}\mathbf{p}^2 (p_0 + mc)^{-2}[\mathbf{S}, [\mathbf{S}, \Phi]] + \frac{1}{2}(p_0 + mc)^{-2}[\mathbf{pS}, [\mathbf{pS}, \Phi]] - \partial_j R^j \Phi. \tag{82}$$

Equation (82) could also be written as

$$\kappa^{-2} p_0 \partial_0 \Phi = \kappa^{-2} \mathbf{p} \nabla_\mathbf{x} \Phi + \tfrac{1}{2} m^2 c^2 \triangle(-A)\Phi - \partial_j R^j \Phi, \tag{83}$$

where $\triangle(-A)$ means $\triangle_A$ with $\mathbf{A} \to -\mathbf{A}$.

If R^j is a function (i.e. a multiple of the unit matrix) and we assume that Φ is a multiple of the unit matrix then the transport and equilibrium equations for Φ are the same as in the spinless case [10]. It follows that we obtain the Jüttner equilibrium distribution [42]

$$\Phi_{EJ} = \exp(-\beta p_0 c) \tag{84}$$

if

$$Y = \left(-\frac{1}{2}\kappa^2 \beta c p_0 p^j + \frac{1}{2}\kappa^2 p^j \right) \frac{\partial}{\partial p^j}. \tag{85}$$

For the modified Jüttner distribution [41, 43]

$$\Phi_{EJ}^M = p_0^{-1} \exp(-\beta p_0 c),\tag{86}$$

we have

$$Y^M = -\frac{1}{2}\kappa^2 \beta c p_0 p^j \frac{\partial}{\partial p^j}.\tag{87}$$

The equilibrium distribution (86) does not depend on the spin variables. We know the magnetized systems which have a non-uniform spin (magnetic moment) distribution. We can obtain such a distribution if $\mathbf{B} = \mathrm{const} \neq 0$ and $\mathbf{E} = 0$. First, let us note that if there is no spin diffusion ($\kappa = 0$), then

$$W_B(\mathbf{p}, \mathbf{n}) = p_0^{-1} \exp(-\beta(p_0 c + \mathbf{B}\mathbf{n}))\tag{88}$$

is the solution of equation (76) with the generator $\mathcal{G}$ defined by the rhs of equation (69). The spin diffusion ($\kappa \neq 0$) also has (86) as the equilibrium distribution if instead of the drift (87) we choose

$$
\begin{aligned}
Y = {}&-\tfrac{1}{2}\kappa^2 \beta c p_0 \mathbf{p}\nabla + mc\beta\kappa^2(p_0 + mc)^{-1}(\mathbf{n}\times\mathbf{B})(\mathbf{p}\times\nabla)\\
&+ \beta\kappa^2\tfrac{1}{2}\mathbf{p}^2 m^{-2}c^{-2}(p_0 + mc)^{-2}((\mathbf{n}\times\mathbf{B})(\mathbf{n}\times\nabla_{\mathbf{n}}) - \mathbf{p}^{-2}\mathbf{p}(\mathbf{n}\times\mathbf{B})\mathbf{p}(\mathbf{n}\times\nabla_{\mathbf{n}})).
\end{aligned}
\tag{89}
$$

Models (88) and (89) may have applications to a description of the time evolution of magnetic properties of relativistic systems with the spin.

10. Discussion

We have discussed an extension of the relativistic diffusion of Schay [5] and Dudley [6] to particles with a spin. The extension is uniquely determined by the rule that similarly as in the spinless case the diffusion should be generated by $M_{\mu\nu}M^{\mu\nu}$. We obtain a diffusion which is the unique relativistic diffusion involving both the momenta and the spin. The diffusion defines the Lindblad dissipative dynamics of particles with spin. This can be an interesting model for a study of the role of spin in the particle motion in the phase space. The relevance of the spin–orbit coupling for dissipation and decoherence has been studied before for the non-relativistic velocities [44, 45]. The spin–orbit coupling results from the relativistic theory. The diffusive dissipation may be relevant to the relativistic description of entanglement and decoherence in the EPR-type experiments [19]. The spin diffusion has been studied before in the theory of magnetism related to the Landau–Lifshitz equation [46]. However, our relativistic approach determines the coupling between spin and momenta in the unique way. It can be useful in a description of dynamic phenomena in the theory of magnetism. The main field of applications should be in the realm of relativistic physics. The dissipation resulting from the synchrotron radiation can be interpreted as a diffusion in a radiation field . The spin diffusion could be compared with experimental results of spin rotation in the synchrotron [47]. The relativistic diffusion of particles without spin has already been applied to heavy ion collisions [48]. For more precise experiments when the results are not averaged over polarizations the description of spin may be relevant. Finally, in astrophysics the problem of propagation of neutrinos scattered in the interstellar medium [49, 50] could be considered as a natural domain

of application of the diffusion equation. As follows from equation (35) for the diffusion of light (treated as spinless in [11]), the spin of the photon does not play any role.

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